Purdue University

Purdue e-Pubs

Department of Computer Science Technical Reports

Department of Computer Science

1976

Piecewise Cubic Hermite Interpolation at the Gaussian Points

Elias N. Houstis
Purdue University, enh@cs.purdue.edu

T. S. Papatheodorou

Report Number:
76-199

PIECEWISE CUBIC HERMITE INTERPOLATION
AT THE GAUSSIAN POINTS

by

E.N. Houstis
Dept. of Computer Sciences
Purdue University
W. Lafayette, IN 47907

and

T.S. Papatheodorou
Dept. of Mathematics
Clarkson College of Tech.
Potsdam, NY 13676

CSD-TR 199

July 1976

PIECEWISE CUBIC HERMITE INTERPOLATION
AT THE GAUSSIAN POINTS

by

E. N. Houstis
Dept. of Computer Sciences
Purdue University
W. Lafayette, Indiana 47907

and

T.S. Papatheodorou
Dept. of Mathematics
Clarkson College of Technology
Potsdam, New York 13676

ABSTRACT

An interpolation scheme based on piecewise cubic polynomials with the Gaussian points as interpolation points is analyzed. Optimal order a priori estimates are obtained for the interpolation error in the maximum norm.

"Piecewise Cubic Hermite Interpolation
at the Gaussian Points"

by

E.N. Houstis and T.S. Papatheodorou

Introduction. We consider an interpolation scheme based on piecewise cubic polynomials with continuous first derivatives and the Gaussian points as interpolation points.

This scheme has been applied as a collocation method by DeBoor and Swartz [2] and Houstis [6] for the numerical solution of ordinary differential equations. Also, Douglas and Dupont [3], [4], [5] and Houstis [7], have studied a collocation method for partial differential equations based on the above scheme.

In sections 1 and 2 we present the formulation for one and two dimensions. In section 3 of this report we obtain optimal order asymptotic estimates for the interpolation error in the L_∞-norm.

1. One-dimensional interpolation scheme. Let $\Delta=(x_i)_1^{N+1}$ be a partition of $I\equiv[a,b], h_i\equiv|x_{i+1}-x_i|, I_i\equiv[x_i,x_{i+1}]$ and $h=\max h_i$. Throughout this report we denote by P_3 the set of polynomials of degree less than 4, and $P_{3,\Delta}$ the set of functions that reduce to polynomials of degree less than 4 in each subinterval $[x_i,x_{i+1}]$. Also we denote by H_Δ the (2N+2)-dimensional vector space of all continuously differentiable piecewise cubic polynomials with respect to Δ. We take $-1<\rho_1,\rho_2<1$ and $w_j>0, j=1,2$ to be the

Gaussian points and weights respectively, so that

$$\int_{-1}^{+1} p(x)\,dx = \sum_{i=1}^{2} p(\rho_i)w_i, \quad p\varepsilon P_3([-1,\ 1]).$$

The Gaussian points and weights in the subinterval $[x_j,\ x_{j+1}]$ are

$$(1.1) \qquad \xi_{2j+i} \equiv \frac{x_j+x_{j+1}}{2} + \rho_i \frac{h_j}{2} \qquad i = 1,2 \quad .$$

We introduce an interpolation operator

$$Q_N : C^1(I) \to H_\Delta$$

such that

$$(1.2) \qquad (Q_N f)(\sigma_\ell) = f(\sigma_\ell), \quad \ell=1,\ldots,2N+2,$$

where $\sigma_1=a$, $\sigma_\ell=\xi_{2j+i}$, $j=1,\ldots,N, i=1,2$, $\sigma_{2N+2}=b$.

This interpolation scheme is well defined. In fact, if $h(x)\varepsilon H_\Delta$ also interpolates f as above, then $e(x)\equiv Q_N f(x)-h(x)$ is a cubic polynomial on $[x_i,x_{i+1}]$, $0\leq i\leq N$ and $e(\sigma_i) = 0$, $1\leq i\leq 2N+2$. We show that e(x) is identically zero in $[x_i,x_{i+1}]$. If this is not so, then without loss of generality we may assume that $e(x)\neq 0$ for all $x\varepsilon[x_1,x_2]$. Rolle's Theorem implies that $e(x_2)D_x e(x_2)>0$. Similarly, $D_x e$ restricted in $[x_2,x_3]$ has roots in (x_2,σ_4), (σ_4,σ_5). Thus, $e(x_3)D_x e(x_3)>0$. By induction

$e(x_{N+1}) D_x e(x_{N+1}) > 0$ contradicting the relation $e(x_{N+1}) = 0$. This proves that $e(x) \equiv 0$ in I.

2. Two-dimensional interpolation scheme. In this section we introduce a two-dimensional analogue of the interpolation scheme of the previous section. Let $\Delta_y = (y_j)_1^{M+1}$ be a partition of $[c,d]$, $J \equiv [c,d]$, $k_j \equiv |y_{j+1} - y_j|$, $J_j \equiv [y_j, y_{j+1}]$ and $k \equiv \max k_j$. Also, we denote by $\rho \equiv \Delta x \Delta y$ a partition of $[a,b] x [c,d]$ and by H_ρ the vector space of all piecewise bicubic polynomials $p(x,y)$ with respect to ρ, such that $D_x^\ell D_y^\eta p(x,y)$ is continuous on $[a,b] x [c,d]$ for all $0 \le \ell, \eta \le 1$.

The Gaussian points and weights in the subinterval $[y_i, y_{i+1}]$ are

$$\tau_{2i+j} \equiv \frac{y_i + y_{i+1}}{2} + \rho_j \frac{k_i}{2} \; , \; j=1, 2 \quad .$$

A two-dimensional interpolation operator is defined as the tensor product

$$Q_\rho \equiv Q_N \otimes Q_M = Q_N Q_M$$

3. Error analysis. In this section, we establish a priori bounds for the interpolation scheme introduced in section 2. For later use, we define the Gramian matrix

$$G_N \equiv \left(B_i(\sigma_j) \; ; \; i,j=1,\ldots,2N+2 \right)$$

of the interpolation operator Q_N. Using the $(2N+2) x (2N+2)$ matrix

$$H_N \equiv \begin{bmatrix} 1 & & & & & \\ h & & & & 0 & \\ & \cdot & & & & \\ & & \cdot & & & \\ & 0 & & \cdot & & \\ & & & & 1 & \\ & & & & h & \end{bmatrix} .$$

we find that

$$H_N^{-1} G_N \equiv \left[\begin{array}{cc|c|c|ccc} 1 & A & & & & & \\ 0 & & & & & & \\ \hline & B & A & & & & 0 \\ \hline & & & \ddots & & & \\ & & & & \ddots & & \\ & 0 & & & & & \\ \hline & & & & B & A & \\ \hline & & & & & B & \begin{matrix} 1 \\ 0 \end{matrix} \end{array}\right]$$

where

$$A \equiv \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} , \qquad B \equiv \begin{bmatrix} \beta & \alpha \\ -\delta & -\gamma \end{bmatrix}$$

and

$$\alpha = \frac{9+4\sqrt{3}}{18} , \; \beta = \frac{9-4\sqrt{3}}{18} , \; \gamma = \frac{3+\sqrt{3}}{36} , \; \delta = \frac{3-\sqrt{3}}{36} .$$

We will also use the matrix

$$T \equiv BA^{-1} = \begin{bmatrix} -7 & 48 \\ 1 & -7 \end{bmatrix} .$$

It is easy to see that for all integers n, ($T^{\circ} \equiv I$),

$$T_n = \begin{bmatrix} a_n & 48c_n \\ c_n & a_n \end{bmatrix}$$

where

$$a_{n+1} = -7a_n + 48c_n \ , \ c_{n+1} = a_n - 7c_n \quad .$$

More generally, from $T^{s+t} = T^s \, T^t$ we get

$$a_{s\pm t} = a_s a_t \pm 48 c_s c_t \ , \ c_{s\pm t} = c_s a_t \pm a_s c_t$$

$$\text{(3.1)} \qquad \begin{aligned} c_s a_t &= \frac{1}{2}\,(c_{s+t} + c_{s-t}) \\ c_s c_t &= \frac{1}{96}\,(a_{s+t} - a_{s-t}) \\ a_s a_t &= \frac{1}{2}\,(a_{s+t} + a_{s-t}) \\ a_{-\ell} &= a_\ell \ , \ c_{-\ell} = -c_\ell \quad . \end{aligned}$$

Let $\lambda_n \equiv |a_n/c_n| = -\,a_n/c_n$. Since $\det(T^n) = 1$, we can easily show that λ_n is decreasing with n and for all n

$$\sqrt{48} < \lambda_n \leqq 7, \ \lambda_1 = 7$$

$$\text{(3.2)} \quad c_n = (-1)^{n+1}|c_n|, \ a_n = (-1)^n |a_n|$$

$$|a_{n+1}| > |a_n| \ , \ |c_{n+1}| > |c_n|$$

Since

$$|a_n| = \frac{1}{2}(|c_{n+1}|-|c_{n-1}|), |c_n| = \frac{1}{96}(|a_{n+1}|-|a_n|)$$

we also have

$$\sum_{\ell=q}^{p} |a_\ell| = \frac{1}{2}(|c_{p+1}|+|c_p|-|c_q|-|c_{q-1}|)$$

(3.3)

$$\sum_{\ell=q}^{p} |c_\ell| = \frac{1}{96}(|a_{p+1}|+|a_p|-|a_q|-|a_{q-1}|) \quad .$$

We introduce a (2N+2)x(2N+2) matrix R in partition form

$$\left[\begin{array}{cc:c:cc}
r_{11} & r_{12} & \cdots & r_{1,2N+1} & r_{1,2N+2} \\
\hdashline
\multicolumn{2}{c:}{R_{11}} & \cdots & \multicolumn{2}{c}{R_{1,N+1}} \\
\multicolumn{2}{c:}{\vdots} & & \multicolumn{2}{c}{\vdots} \\
\multicolumn{2}{c:}{R_{N,1}} & \cdots & \multicolumn{2}{c}{R_{N,N+1}} \\
\hdashline
r_{2N+2,1} & r_{2N+2,2} & \cdots & r_{2N+2,2N+1} & r_{2N+2,2N+2}
\end{array}\right]$$

where the first and last rows are defined as

$$[r_{1,2j-1},\ r_{1,2j}] \equiv \frac{(-1)^{j+1}}{c_N}\ [c_{N-j+1}\ \ a_{N-j+1}]$$

$$j=1,\ldots,N+1$$

$$[r_{2N+2,2j-1},\ r_{2N+2,2j}] \equiv \frac{(-1)^{N-j}}{c_N}\ [-c_{j-1}\ \ a_{j-1}]$$

while the 2x2 matrices $R_{n,m}$ are defined as

$$R_{n,m} \equiv A^{-1}[(-T)^{n-1} Z_m + \sigma_{n,m}(-T)^{n-m}],$$
$$n=1,\ldots,N \quad , \quad m=1,\ldots,N+1$$

with

$$Z_1 \equiv \begin{bmatrix} 0 & \lambda_N \\ 0 & 1 \end{bmatrix} \quad , \quad Z_m \equiv \frac{(-1)^m}{c_N} \begin{bmatrix} c_{N-m+1} & a_{N-m+1} \\ 0 & 0 \end{bmatrix} \qquad m=2,\ldots,N+1$$

and

$$\sigma_{n,m} = \begin{cases} 1 & \text{if } 2 \leq m \leq n \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.1. The matrix $H_N^{-1}\ G_N$ is invertible and its inverse is the matrix R.

Proof: Let $S \equiv R(H_N^{-1}G_N)$. It is enough to show that S=I. We partition S into blocks:

$$S = \begin{bmatrix} s_{11} & \tau_{11} & \cdots & \tau_{1N} & s_{1,2N+2} \\ \omega_{11} & S_{11} & \cdots & S_{1N} & \omega_{1,2N+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \omega_{N1} & S_{N1} & & S_{NN} & \omega_{N,2N+2} \\ s_{2N+2,1} & \tau_{2N+2,1} & \cdots & \tau_{2N+2,N} & s_{2N+2,2N+2} \end{bmatrix}$$

where each s_{ij} is 1x1, S_{ij} is 2x2, ω_{ij} is 2x1 and τ_{ij} is 1x2.

Performing the multiplication of the matrices R and $H_N^{-1}\ G_N$ we obtain

$$s_{11} = r_{11} = 1$$

$$\begin{aligned} \tau_{ij} &= [s_{1,2j}\quad s_{1,2j+1}] = [r_{1,2j-1}\quad r_{1,2j}]\ A+[r_{1,2j-1}\quad r_{1,2j+2}]B \\ &= \frac{(-1)^j}{c_N}\{[c_{N-j+1}\ a_{N-j+1}]-[c_{N-j}a_{N-j}]T\}A \\ &= \frac{(-1)^j}{c_N}\{[c_{N-j+1}\ a_{N-j+1}]-[c_{N-j+1}\ a_{N-j+1}]\}A \\ &= [0,\ 0] \end{aligned}$$

and

$$s_{1,2N+2} = r_{1,2N+1} = 0 \quad .$$

Similarly

$$\omega_{i,1} = \omega_{i,2N+2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \tau_{2N+2,j} = [0 \; 0], \; i,j = 1,\ldots,N$$

and

$$s_{2N+2,1} = 0 \quad , \; s_{2N+2,2N+2} = 1.$$

For the square blocks $S_{n,m}$ we find

$$S_{n,m} = R_{n,m} A + R_{n,m+1} B$$

$$= A^{-1}(-T)^{n-1}\{Z_m + Z_{m+1} T + (\sigma_{n,m}-\sigma_{n,m+1})(-T)^{1-m}\} A$$

From the definition of Z_m and T we obtain $Z_m + Z_{m+1} T = \delta_1^m I$.

Then from the definition of $\sigma_{n,m}$ we get

$$S_{n,m} = \delta_n^m I.$$

This concludes the proof of Lemma 3.1.

Lemma 3.2. If G_N is the Grammian of the interpolation operator Q_N then

(3.4) $\quad ||(H_N^{-1} G_N)^{-1}||_\infty < 100$

for all $N \geq 2$.

Proof. Let

$$||R||_\ell \equiv \Sigma_{m=1}^{2N+2} |r_{\ell m}| \quad .$$

From the definition of R and relations (3.1), (3.2), (3.3), we obtain

$$||R||_1 \equiv \Sigma_{j=1}^{N}(|r_{1,2j+1}|+|r_{1,2j}|)$$

$$= \frac{1}{|c_N|}\Sigma_{j=1}^{N}(|c_{N-j+1}|+|a_{N-j+1}|)$$

$$= \frac{1}{|c_N|}\Sigma_{\ell=1}^{N}(|c_\ell|+|a_\ell|)$$

$$\leqq \frac{7}{12}\frac{|a_N|}{|c_N|}+\frac{9}{2}+\frac{5}{15}\frac{|a_1|}{|c_N|}-\frac{7}{2}\frac{|c_1|}{|c_N|}$$

$$\leqq 23/2 \quad .$$

It is easy to see that $||R||_{2N+2} = ||R||_1$. For the remaining rows we use (3.1) (3.2) to get that for $2 \leqq m \leqq n$

$$AR_{n,m} = (-T)^{n-1} z_m + \sigma_{n,m} (-T)^{n-m}$$

$$= \frac{1}{2|c_N|}\begin{bmatrix} -|c_{N-n-m+2}|+|c_{N-n+m}| & |a_{N-n-m+2}|+|a_{N-n+m}| \\ \frac{1}{48}(-|a_{N-n-m+2}|+|a_{N-n+m}|) & |c_{N-n-m+2}|+|c_{N-n+m}| \end{bmatrix}$$

while for $n < m$

$$AR_{n,m} = \frac{1}{2|c_N|} \begin{bmatrix} -|c_{N+n-m}|+|c_{N-n-m+2}| & |a_{N+n-m}|+|a_{N-n-m+2}| \\ \\ \frac{1}{48}\,(|a_{N+n-m}|-|a_{N-n-m+2}|) & |c_{N+n-m}|-|c_{N-n-m+2}| \end{bmatrix}$$

Finally, for m=1

$$AR_{n,1} = \frac{1}{|c_N|} \begin{bmatrix} 0 & |a_{N-n+1}| \\ 0 & |c_{N-n+1}| \end{bmatrix}$$

Using again the relations (3.1) through (3.3) we now find

$$\sum_{m=1}^{N} ||AR_{n,m}||_1 \leqq$$

$$\frac{1}{2}\left[2\,\frac{|a_{N-n+1}|}{|c_N|} + \frac{1}{96}\,\left(\frac{|a_{N-n+1}|}{|c_N|} + \frac{|a_{N-n}|}{|c_N|} + \frac{|a_N|}{|c_N|} + \frac{|a_{N-1}|}{|c_N|}\right)\right.$$

$$\left. + \frac{1}{2}\,\left(\frac{|c_{N-n}|}{|c_N|} + 9 + \frac{|c_{N-1}|}{|c_N|} + \frac{|a_N|}{|c_N|}\right)\right] \leqq \frac{35}{3}$$

and

$$\sum_{m=1}^{N} ||AR_{n,m}||_2 \leqq$$

$$\frac{1}{2}\left[2\,\frac{|c_{N-n+1}|}{|c_N|} + \frac{1}{96}\left(\frac{|a_{N-n}|}{|c_N|} + 9\,\frac{|a_N|}{|c_N|} + 48 + \frac{|c_{N-n+1}|}{|c_N|}\right.\right.$$

$$\left.\left. + \frac{|c_{N-n}|}{|c_N|} + \frac{|a_{N-1}|}{|c_N|} + \frac{|a_n|}{|c_N|} + \frac{|c_{N-1}|}{|c_N|}\right)\right] \leqq 2 \quad .$$

By definition now, we have for $\ell=1,2$

$$||R||_{2n+\ell} = \sum_{m=1}^{N} ||A^{-1}AR_{n,m}||_\ell \leqq \sum_{m=1}^{N} ||A^{-1}||_\infty ||AR_{n,m}||_\ell$$

while

$$||A^{-1}||_\infty = \frac{7\sqrt{3}+9}{4} \quad .$$

Thus, for the norm $||R||_\infty = \max_i ||R||_i$ the following bound holds

$$||R||_\infty = ||(H^{-1}G_N)^{-1}||_\infty < 100.$$

This concludes the proof of Lemma 3.2.

Remark. As the proof of Lemma 3.2 suggests the bound (3.4) can be improved. Our conjecture is that a more careful analysis will show that the norm $||(H_N^{-1}G_N)^{-1}||_\infty$ is decreasing in N, that

$$\lim_{N\to\infty}||(H_N^{-1}G_N)^{-1}||_\infty = \frac{69-29\sqrt{3}}{2}$$

and that for all $N \geq 2$

$$\frac{69-29\sqrt{3}}{2} \leqq ||(H_N^{-1}G_N)^{-1}||_\infty \leqq ||H_2^{-1}G_2|| = \frac{33\sqrt{3}+9}{7}$$

Numerical experiments confirm this conjecture.

Lemma 3.3. The Gramian matrix G_N of the interpolation operator Q_N is nonsingular and

$$(3.5) \qquad ||G_N^{-1}||_\infty \leqq 100\ N$$

for all $N \geq 2$.

Proof. (3.5) follows easily from Lemmata 3.1 and 3.2.

Lemma 3.4. Let Q_N be the interpolation operator defined by (2.2). Then, (i) Q_N is a linear projector on $C^1(I)$ with range H_Δ and (ii) there exists a constant c such that $||Q_N|| \leqq c\ N$.

Proof. Conclusion (i) follows easily from Lemma 3.3. To prove (ii) let Λ be the dual space of H_Δ and $\{B_i\}_{i=1}^{2N+2}$ and $\{\delta_{\sigma_i}\}_{i=1}^{2N+2}$ be bases for H_Δ and Λ, where δ_{σ_i} are the point evaluation functionals. Using [1, Prop. 3] one may easily show that

$$||Q_\Delta|| \leqq \max_{a\varepsilon R^n} \frac{||\Sigma_i a_i B_i||_\infty}{||a||} ||(\delta_{\sigma_i} B_j)^{-1}||_\infty \max_i ||\delta_{\sigma_i}||$$

$$\leqq 2\ ||G_N^{-1}||_\infty \leqq cN$$

where, $G_N = (\delta_{\sigma_i} B_j)$ and, by (3.5), $c = 200$. This concludes the proof of Lemma 3.4.

Theorem 3.1. If $f\varepsilon W^{4,\infty}(I)$, then

(i) $Q_N f \rightarrow f$, as $N \rightarrow \infty$

and

(ii) for the interpolation error we have

$$||Q_N f - f||_\infty \leqq ch^4$$

where c is independent of h.

Proof. Let $\partial_H f$ be the Hermite interpolant of f, defined by interpolation of f and its first derivative at the nodes of the partition Δ. From the triangle inequality we find

$$(3.6) \quad ||f-Q_N f||_\infty \leqq (1+||Q_N||)||f-\partial_H f||_\infty \quad .$$

Moreover, for the Hermite interpolation error, it is known [10, Thrm 3.6]

(3.7) $$||f - \partial_H f||_\infty \leq \frac{1}{384} h^4 ||D^4 f||_\infty .$$

From (3.6), (3.7) and Lemma 3.4, we now get

$$||f - Q_N f||_\infty = O(h^3).$$

This proves conclusion (i).

Also, Theorem 2 [9, p. 251] and conclusion (i) imply that there is a constant K, independent of N, such that

(3.8) $$||Q_N|| \leq K \qquad N=2,3,\ldots$$

From (3.6) and (3.8) conclusion (ii) follows.

Theorem 3.2. If $f \in W^{4,\infty}(I \times J)$ then for the interpolation error we have

$$||Q_\rho f - f||_\infty \leq c|\rho|^4$$

where $|\rho| = \max(h,k)$ and c is a constant independent of h and k.

Proof. From (3.6)-(3.8) and the triangle inequality we have

$$\begin{aligned} ||f-Q_\rho f||_\infty &\leq ||f-Q_N f||_\infty + ||Q_N f - Q_N Q_M f||_\infty \\ &\leq ||f-Q_N f||_\infty + ||Q_N|| \; ||f-Q_M f||_\infty \\ &\leq c\,(h^4+k^4) \leq c|\rho|^4 \end{aligned}$$

which concludes the proof of the theorem.

4. <u>Numerical results</u>. In this section we present some numerical results concerning the approximation of the functions e^x, x^4 by interpolation at Gaussian points with the space H_Δ. These results indicate that the interpolation scheme introduced at Section 2 is fourth-order accurate in the L_∞-norm. The partition Δ used is uniform with mesh length $h = 1/N$. The rate of convergence estimate

$$-\log \left(\frac{\text{error for } h}{\text{error for } h/2}\right)/\log 2$$

is also given.

N	$\|\|e^x - Q_N e^x\|\|_\infty$	Convergence Rate
3	3.106×10^{-5}	
6	2.325×10^{-6}	3.74
12	1.646×10^{-7}	3.8
24	1.096×10^{-8}	3.9
48	7.070×10^{-10}	3.95

N	$\|\|x^4 - Q_N x^4\|\|_\infty$	Convergence Rate
3	4.155×10^{-4}	
6	2.678×10^{-5}	3.96
12	1.674×10^{-6}	3.99
24	1.047×10^{-7}	4.00
48	6.541×10^{-9}	4.00

REFERENCES

1. Carl DeBoor, Bounding the error in spline interpolation, SIAM review, 10 (1974), pp. 531-544.

2. C. DeBoor and B. Swartz, Collocation at Gaussian points, SIAM J. Numer. Anal., 10 (1973), pp. 582-606.

3. Jim Douglas, Jr. and Todd Dupont, A finite element collocation method for quasilinear Panabolic Equations, Math. Comp., 27 (1973), pp. 212.

4. Jim Douglas, Jr. and Todd Dupont, A super convergence result for the approximate solution of the heat equation by a collocation method, in Mathematical Foundations of Finite Element Method with Applications to Partial Differential Equations, A. K. Aziz, Editor, Academic Press, New York, 1972.

5. Jim Doublas, Jr. and Todd Dupont, Collocation methods for panabolic equations in a single space variable based on C^1-piecewise polynomial spaces, Springer Lecture Note Series, Vol. 385, Springer-Verlog, Berlin, 1974.

6. E. N. Houstis, A. collocation method for systems of non-linear ordinary differential equations, to be published in the Journal of Mathematical Analysis and Applications.

7. E. N. Houstis, Application of method of collocation on lines for solving nonlinear hyperbolic problems, to be published in the Journal of Mathematics of Computation.

8. M. A. Krasnosel'skii, G. M. Vainikko, P. P. Zabreiko, Yu. B. Rutitskii, V. Ya Stetsenko, Approximate solution of operator equations, Wolfers-Noordhoff, 1969.

9. L. V. Kantoronich and G. P. Akilov, Functional analysis in normed spaces, Pergamon Press, 1969. (English translation).

10. M. H. Schultz, Spline Analysis, Prentice-Hall, 1973.